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CONTROL OF THE EVOLUTION OF A DYNAMICAL SYSTEM UNDER HIGH-FREQUENCY EXCITATIONS†

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A controlled dynamical system subjected to high-frequency excitations is investigated. A standard controlled system is constructed using a change of variables, which generalizes the Bogolyubov change of variables in the problem of a pendulum with a vibrating suspension point. An effective procedure is developed for the approximate solution of the problem of optimal control over an asymptotically large range of variation of the argument. The property of closeness of the approximate solution to the exact solution with respect to the slow variable and functional is established. A generalization of the algorithm for solving the problem to dynamical systems with variable parameters is given. The effectiveness of the approach is illustrated by investigating the problems of control of mechanical systems: the oscillations and rotations of a rigid body with a vibrating axis, and the motion of a "microparticle" in a force field, modelled by travelling and standing waves.

1. FORMULATION OF THE PROBLEM

A controlled dynamical system, subjected to high-frequency periodic excitations, is considered [1]. It is assumed that the equations of motion (for example, in Lagrange form) can be represented in terms of dimensionless quantities as follows

$$\begin{gathered} \ddot{q} = Q(\theta, q, \dot{q}, u, \lambda), \quad q(t_0) = q^0, \quad \dot{q}(t_0) = \dot{q}^0 \\ \theta = \lambda t, \quad t_0 \le t \le T, \quad \lambda \gg 1, \quad u \in U, \quad q \in D_q, \quad \dot{q} \in D_{\dot{q}} \end{gathered} \tag{1.1}$$

Here a dot denotes a derivative with respect to time, $(q, \dot{q})$ is a $2n$-vector of the phase variables, θ is the fast phase of the external periodic excitation, u is the r-vector of control, and U is a fixed set. The numerical parameter λ can take asymptotically large values ($\lambda \to \infty$), i.e. $\varepsilon = \lambda^{-1}$ is a small parameter. It is assumed that the quantities $|q|$, $|\dot{q}|$, $|u|$ are of the order of unity with respect to the large parameter λ. The quantities t_0, q^0, $\dot{q}^0$ are assumed to be given, and the controlled motion of the system (1.1) is considered in a fixed time interval $t_0 \le t \le T$ (of the order of unity). The function Q must be 2π-periodic and piecewise-continuous in θ; it is assumed to be fairly smooth enough with respect to the remaining arguments. The structural characteristics and properties of smoothness of the functions Q and u will be refined below. They are needed to enable the problem of control and optimization for system (1.1) to be reduced to a standard form, allowing of the use of asymptotic methods [1–3].

We will formulate the optimal control problem. We will assume that the final conditions imposed on the variables q and $\dot{q}$ at a fixed instant of time $t = T$ have the general form

$$M(q, \dot{q})|_T = 0, \quad M = (M_1, \dots, M_m), \quad 0 \le m \le 2n \tag{1.2}$$

In particular, conditions (1.2) may be absent ($m = 0$), or correspond to the two-point problem: $q(T) = q^T$, $\dot{q}(T) = \dot{q}^T$, when q^T, $\dot{q}^T$ are known. The vector function M is assumed to be smooth enough in the region under consideration, and a possible regular dependence on λ, for example a smooth dependence on ε, $|\varepsilon| \le \varepsilon_0$, is not indicated for brevity.

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The performance index of the control is taken in the form of the integral functional

$$J[u] = g(q, \dot{q})|_T + \int_{t_0}^{T} G(\theta, q, \dot{q}, u)dt \rightarrow \min_u, \quad u \in U \tag{1.3}$$

The dependences of the functions g and G on the arguments are similar to those indicated above for the functions M and Q respectively. For a fixed value of λ, relations (1.1)–(1.3) give the standard formulation of the problem of optimal control over a fixed time interval, the solution of which is constructed based on the necessary conditions in the form of the maximum principle [4].

Note that the dependence of the functions Q and G on the fast phase considerably complicates both the analytical and numerical investigation of the optimal control problem because of the oscillations of the right-hand sides of the equations with time for as high a value of the frequency λ as desired. However, when certain natural conditions are satisfied, we can use this property to construct an approximate solution by reducing the equations of the boundary-value problem of the maximum principle to a Bogolyubov standard form and employ the method of averaging [5, 6] using the well-known method [2, 3]. These conditions are related to the possibility of using a change of variables [1], which generalizes the Bogolyubov change of variables in the classical problem of the oscillations of a plane pendulum with a vertically vibrating suspension point [5].

2. REDUCTION OF THE CONTROL PROBLEM TO A STANDARD FORM

We will first make an elementary change of the time argument t for the fast phase $\theta = \lambda t$, which varies over an asymptotically long interval $\Delta\theta = \lambda\Delta t \sim \varepsilon^{-1}$. Relations (1.1) then take the form

$$\begin{gathered} q'' = \varepsilon^2 Q(\theta, q, \varepsilon^{-1}q', u, \varepsilon^{-1}), \quad q(\theta_0) = q^0, \quad q'(\theta_0) = \varepsilon\dot{q}^0 \\ \theta_0 = \lambda t_0, \quad \Theta = \lambda T \gg 1, \quad \theta_0 \le \theta \le \Theta \end{gathered} \tag{2.1}$$

Here the prime denotes a derivative with respect to the phase θ. Note that the initial value of the derivative $q' \sim \varepsilon \ll 1$. The main requirement for the change of variables related to the structure of the function Q is that the variable q be slow over the interval $\Delta\theta$ (2.1) under consideration.

Following the well-known approach [1–5], we will assume that the right-hand side of Eq. (2.1) has the form

$$\varepsilon^2 Q \equiv \varepsilon R(\theta, q, q') + \varepsilon^2 S(\theta, q, \varepsilon^{-1}q', u) \tag{2.2}$$

Here R and S are 2π-periodic piecewise-continuous functions of θ, which depend regularly on these arguments in the region $q \in D_q$, $\varepsilon^{-1}q' \in D_{\dot{q}}$, $u \in U$. The possible regular dependence on the small parameter ε is not indicated for brevity.

The class of functions Q (1.1) or (1.2), which satisfy condition (2.2), is fairly wide. In applied problems (systems with rapidly rotating phase [5]) we usually have the situation where $R \equiv 0$, i.e. $Q = S$. According to representation (2.2) for $R \equiv 0$, the dependence $Q \sim \lambda$ is allowed, which considerably extends the class of functions Q [1, 5].

For system (2.1), (2.2) to be representable in a standard form with a small parameter ε in the interval $\Delta\theta \sim \varepsilon^{-1}$, it is sufficient to require that the following condition should be satisfied

$$\langle R_0 \rangle \equiv 0, \quad R_0 = R(\theta, q, 0), \quad \theta \ge \theta_0, \quad q \in D_q \tag{2.3}$$

where the angle brackets denote averaging over θ. We can then propose the replacement $(q, q') \rightarrow (x, y)$, in which both the n-vectors x and y are slow: $x' \sim \varepsilon$, $y' \sim \varepsilon$. In fact, we assume

$$\begin{gathered} q = x + \varepsilon R^{**}(\theta, x), \quad q' = \varepsilon y + \varepsilon\Delta R^*(\theta, x); \quad R^* = \int_{\theta_0}^{\theta} R_0(\tau, x)d\tau, \quad R^{**} = \int_{\theta_0}^{\theta} \Delta R^*(\tau, x)d\tau \\ x \in D_q, \quad [y + \Delta R^*(\theta, x)] \in D_{\dot{q}}, \quad \Delta R^* \equiv R^* - \langle R^* \rangle \end{gathered} \tag{2.4}$$

The replacement $q \to x$ (2.4) is close to identical and is independent of y; the relation between q' and y depends on x and assumes that q' is small ($q' \sim \varepsilon; x, y \sim 1$). According to condition (2.3), the functions R^* and R^{**} will be 2π-periodic in θ and bounded for all $\theta \geq \theta_0, x \in D_q$. It is essential that the change of variables (2.4) does not involve the unknown function u.

With the condition that the vector function R_0 is continuous in q, we will differentiate relations (2.4) by virtue of system (2.1) and (2.2) to obtain a system of two vector equations of standard form [2, 3]

$$\begin{aligned} &x' = \varepsilon X(\theta, x, y, \varepsilon), \quad x(\theta_0) = q^0; \quad y' = Y(\theta, x, y, u, \varepsilon), \quad y(\theta_0) = \dot{q}^0 \\ &X \equiv [I + \varepsilon R_x^{**\prime}(\theta, x)]^{-1} y = y - \varepsilon R_x^{**\prime} y + \varepsilon^2 \ldots \\ &Y \equiv -\Delta R_x^{*\prime}(\theta, x) X + \varepsilon^{-1}[R(\theta, x + \varepsilon R^{**}, \varepsilon y + \varepsilon \Delta R^*) - R_0(\theta, x)] + \\ &+ S(\theta, x + \varepsilon R^{**}, y + \Delta R^*, u) = -\Delta R_x^{*\prime} X + R'_{0x} R^{**} + (R'_{q'})_0 (y + \Delta R^*) + S + \varepsilon \ldots \end{aligned} \tag{2.5}$$

The functions X and Y (2.5) are assumed to be smooth in x, y and u in the above-mentioned ranges when $\varepsilon > 0$ is sufficiently small. The function X is linear in y, and in the first approximation $X = X_0 = y$; moreover, it is independent of u. The function $Y \approx Y_0$ is quadratic in R. This is the essential structural difference between system (2.5) and the previously investigated weakly controlled systems of standard form [2, 3].

The slow variable x is close to q with an error $O(\varepsilon)$ in the interval $\Delta\theta \sim \varepsilon^{-1}$, i.e. $\Delta t \sim 1$. The slow variable y differs by an amount $\Delta R^* \sim 1$ from the velocity $\dot{q}$, which is not slow in the common sense [5, 6] and, moreover, $\ddot{q} \sim \varepsilon^{-1}$. However both vectors q and $\dot{q}$ are defined with the required degree of accuracy (with an error $O(\varepsilon)$).

Note that if $R \equiv 0$, the transformations of (2.4) become elementary, $q = x$ and $q' = \varepsilon y$, and system (2.5) takes the form

$$x' = \varepsilon y, \quad x(\theta_0) = q^0; \quad y' = \varepsilon S(\theta, x, y, u), \quad y(\theta_0) = \dot{q}^0 \tag{2.6}$$

In the general case, a system of the form $\ddot{\varphi} = \varepsilon\Phi$ is a multiphase system (the function Φ is periodic in φ) and presents considerable difficulties for an asymptotic analysis [7]. A local investigation can usually be carried out assuming that $\dot{\varphi} = \sqrt{\varepsilon}\omega$ in the time interval $\Delta t \sim 1/\sqrt{\varepsilon}$ [6].

A system of equations of the form (2.1), (2.2) can be the initial one and is obtained without transforming the argument $\theta = \lambda t$, assuming the asymptotic unboundedness of the parameter λ. This indicates the relative smallness of the generalized forces $\varepsilon R + \varepsilon^2 S$ when $q \sim 1, \dot{q} \sim \varepsilon, \lambda = 1$.

We will transform the final conditions (1.2) using relations (2.4) to obtain

$$M(x + \varepsilon R^{**}(\theta, x), y + \Delta R^*(\theta, x))|_\Theta = 0, \quad \Theta = T\varepsilon^{-1} \tag{2.7}$$

In the first approximation in the small parameter, the term εR^{**} in the first argument of the function M (2.7) can be ignored. The presence of the term ΔR^* in the second argument requires that the quantities t_0 and $T \sim 1$ should be specified with high accuracy (an error of $O(\varepsilon^2)$). A change in the parameters t_0 and T by $O(\varepsilon)$ leads to a substantial variation of the quantity ΔR^* in (2.7), of an order of unity for $\theta = \Theta$. As was pointed out, this property is due to the singular nature of the variable $\dot{q}$, since $\ddot{q} \sim \varepsilon^{-1}$.

We will write functional (1.3) in the new variables

$$J[u] = g(x + \varepsilon R^{**}, y + \Delta R^*)|_\Theta + \varepsilon \int_{\theta_0}^{\Theta} G(\theta, x + \varepsilon R^{**}, y + \Delta R^*, u) d\theta \tag{2.8}$$

in which the functions R^{**} and ΔR^* are defined by (2.4), as was the case for formula (2.7). To find the solution in the first approximation, we can confine ourselves to quantities of an order of unity, while the term εR^{**} in g and G can be dropped.

The functions M, g and G may depend regularly on the small parameter ε, which, in the first approximation considered later, is assumed to be equal to zero.

We will briefly describe the asymptotic procedure of the approximate solution of the terminal problem of optimal control (2.5)–(2.8) using the method described previously in [2, 3].

3. THE USE OF THE AVERAGING METHOD

We will drop the terms $O(\varepsilon^2)$ in Eqs (2.6), i.e. terms $O(\varepsilon)$ in the expressions for X and Y; we have

$$\begin{aligned} &x' = \varepsilon y, \quad x(\theta_0) = q^0; \quad y' = \varepsilon Y_0(\theta, x, y, u), \quad y(\theta_0) = \dot{q}^0 \\ &Y_0 \equiv -\Delta R_x^{*\prime} y + R'_{0x}R^{**} + (R'_{q'})_0(y + \Delta R^*) + S(\theta, x, y + \Delta R^*, u) \end{aligned} \tag{3.1}$$

Note that the function Y_0 is quadratic in R, R^{**} and ΔR^*.

System (3.1) can be represented in the form of a second-order equation in x and contains a singular dependence on x'

$$x'' = \varepsilon^2 Y_0(\theta, x, \varepsilon^{-1}x', u), \quad x(\theta_0) = q^0, \quad x'(\theta_0) = \varepsilon\dot{q}^0 \tag{3.2}$$

However, on transferring to the argument t, the equation does not contain this singularity

$$\ddot{x} = Y_0(\lambda t, x, \dot{x}, u), \quad x(t_0) = q^0, \quad \dot{x}(t_0) = \dot{q}^0 \tag{3.3}$$

The right-hand side of Eq. (3.3) is a rapidly oscillating function of t. Note that the structures of Eqs (3.2) and (3.3), and Eqs (2.1), (2.2) and (1.1) corresponding to them, differ considerably.

Henceforth, in the final conditions (2.7) and the control performance index (2.8) we will drop quantities $O(\varepsilon)$. Using the mathematical techniques of the maximum principle [4] we will write the necessary optimality conditions for the control u

$$\begin{aligned} &H = \varepsilon H_0(\theta, x, y, p, u) \to \max_u, \quad u \in U; \quad p = (p_x, p_y)^T \\ &H_0 = (p_x, y) + (p_y, Y_0) - G, \quad u = u^*(\theta, x, y, p) \\ &p'_x = -\varepsilon(H'_{0x})^*, \quad p'_y = -\varepsilon(H'_{0y})^*, \quad (H_0)^* = H_0|_{u^*}, \quad p_{x,y}(\Theta) = (\chi, M'_{x,y})_\Theta - g'_{x,y}|_\Theta \end{aligned} \tag{3.4}$$

Here H is the Hamilton function of the problem, p_x and p_y are variables (momenta) adjoint to x and y respectively, and χ is the m-vector of the Lagrange multipliers.

We will assume that the vector function u^* is defined explicitly in analytical form and is 2π-periodic in θ and, of course, smooth in x, y and p. After substituting the expression for u^* into Eqs (3.1) and (3.4) we obtain a $2n$-dimensional Hamiltonian system of equations in Bogolyubov's standard form. It is required to construct a solution of the boundary-value problem of the maximum principle, including obtaining the m-vector of the Lagrange multipliers χ, with an error $O(\varepsilon)$. The problem of a numerical-analytical investigation can be simplified considerably using the averaging method [5, 6] and the asymptotic procedure described in [2, 3].

In the first approximation in ε, we obtain the averaged boundary-value problem in the original "slow" time t

$$\begin{aligned} &\dot{\xi} = \eta, \quad \xi(t_0) = q^0; \quad \dot{\eta} = \mathbf{H}(\xi, \eta, \psi), \quad \eta(t_0) = \dot{q}^0; \quad \psi = (\psi_\xi, \psi_\eta)^T \\ &\dot{\psi}_\xi = -h'_\xi, \quad \dot{\psi}_\eta = -h'_\eta, \quad N_{\xi,\eta} = \psi_{\xi,\eta}(T) - (\chi, M'_{\xi,\eta})_T + q'_{\xi,\eta}|_T = 0 \\ &M\left(\xi, \eta + \Delta R^*\left(\frac{t}{\varepsilon}, \xi\right)\right)\Big|_T = 0, \quad t_0 \le t \le T \end{aligned} \tag{3.5}$$

Note that the function ΔR^* depends 2π-periodically on $\theta = t\varepsilon^{-1}$.

System (3.5) can be integrated in the short interval $\Delta t \sim 1$; it is autonomous and Hamiltonian. The averaged variables ξ, η and $\psi_{\xi,\eta}$, corresponding to x, y and $p_{x,y}$ are defined by the Hamiltonian H_0^* (3.4), i.e.

$$\begin{aligned} &h(\xi, \eta, \psi) \equiv \langle H_0^*(\theta, \xi, \eta, \psi)\rangle = (\psi_\xi, \eta) + (\psi_\eta, \langle Y_0^*\rangle) - \langle G^*\rangle, \\ &h'_\psi = (\eta, \mathbf{H})^T, \quad \mathbf{H}(\xi, \eta) = \langle Y_0^*\rangle \end{aligned} \tag{3.6}$$

The first integral h = const (3.6) of system (3.5) can be used for the analytical integration of the equations or for a numerical solution of the problem for monitoring the accuracy (or computational errors). The investigation can be simplified considerably in the case of a linear or linear-quadratic control problem with periodic rapidly oscillating coefficients. For non-linear problems, for which one can carry out operations to maximize the function H_0 with respect to u (3.4) and averaging (3.6) in analytical form, one can also achieve a considerable simplification of the numerical-analytical solution of boundary-value problem (3.5). A more general case requires the use of numerical methods for both maximization and averaging, which leads to extremely cumbersome procedures. The use of the averaging procedure is justified by the possibility of cancelling a number of terms of Eqs (2.5) and considerably shortening the integration interval.

4. BASIC RESULTS

Formally, the scheme for solving the boundary-value problem consists of finding the unknown initial data ψ_ξ^0, ψ_η^0 and the parameter χ, which define the desired variables ξ, η, ψ_ξ, ψ_η as the solution of a Cauchy problem. We will represent it in the form

$$\begin{aligned} \xi &= \xi(t-t_0, q^0, \dot{q}^0, \psi^0, \chi), \quad \eta = \eta(t-t_0, q^0, \dot{q}^0, \psi^0, \chi) \\ \psi &= \psi(t-t_0, q^0, \dot{q}^0, \psi^0, \chi), \quad \psi^0 \in \Psi^0, \quad \chi \in X \end{aligned} \tag{4.1}$$

The possible dependence on θ_0 is not indicated for brevity. The sets Ψ^0 and X are chosen from conditions (1.1), imposed on $q, \dot{q}$ and u, taking into account the replacement formulae (2.4). To determine the $2n$-vector ψ^0 and the m-vector χ ($2n + m$ unknowns in total) one uses the $2n + m$ final conditions (3.5) with $t = T$, into which we substitute expressions (4.1)

$$\begin{aligned} &N^*(T-t_0, \Theta, \theta_0, q^0, \dot{q}^0; \psi^0, \chi) = 0, \quad q^0 \in D_q, \quad \dot{q}^0 \in D_{\dot{q}} \\ &M^*(T-t_0, \Theta, \theta_0, q^0, \dot{q}^0; \psi^0, \chi) = 0 \end{aligned} \tag{4.2}$$

The $2n$-vector function N^* is obtained from the transversality conditions (3.5) for ψ_ξ, ψ_η by substituting expressions (4.1) with $t = T$. The above-mentioned 2π-periodic dependence on θ_0 and Θ (in terms of the functions ΔR^* and R^{**}) is also indicated. The expression for M^* (4.2) is obtained from the final condition $M = 0$ (3.5) on substituting the solution (4.1).

It is further required to obtain the real roots ψ^*, χ^* as a function of known (measured) quantities, to which the time parameters $T - t_0$, Θ, θ_0 and the phase parameters q^0, $\dot{q}^0$ are related. Suppose these roots have been obtained; then by substituting ψ^*, χ^* into expressions (4.1) and then (2.4), (3.4), (2.7) and (2.8) we can determine the approximate phase trajectory q_0 and $\dot{q}_0$, the open-loop control u_0, and the discrepancy M_0 in satisfying the final conditions and the value of the function J_0. The following assertions hold.

Theorem 1. When the conditions $q_0 \in D_q$, $\dot{q}_0 \in D_{\dot{q}}$ hold and the roots ψ^*, χ^* are simple, i.e. $\det(\partial(N^*, M^*)/\partial(\psi^*, \chi^*)) \neq 0$ in the region of the known parameters under consideration, the approximate solution determined above is ε-close to the optimal solution in terms of the trajectory, the final condition and the functional. The closeness in terms of the control occurs in the sense of the integral metric, this closeness being uniform if the function u^* of (3.4) is smooth.

The proof of the theorem follows from the constructions given and from the asymptotic methods of optimal control [2, 3].

We will derive expressions for the approximate solution of the optimal control problem (1.1)–(1.3) in the original variables. According to relations (2.4) and (4.1) we obtain the phase trajectory

$$\begin{aligned} q_0 &= \xi(t-t_0, q^0, \dot{q}^0, \psi^*, \chi^*) \equiv q_0(t-t_0, T-t_0, \Theta, \theta_0, q^0, \dot{q}^0) \\ \dot{q}_0 &= \eta(t-t_0, q^0, \dot{q}^0, \psi^*, \chi^*) + \Delta R^*(\theta, q_0) \equiv \dot{q}_0(t-t_0, T-t_0, \Theta, \theta_0, q^0, \dot{q}^0) \end{aligned} \tag{4.3}$$

An important feature is the dependence of q^0 and $\dot{q}^0$ on Θ, $\theta_0 \sim \varepsilon^{-1}$; it disappears when $R_0 \sim \varepsilon$. Similar properties occur in problems of the weak control of quasi-linear oscillatory systems over an asymptotically long time interval [1, 3], when the terminal term in the functional and the final conditions depend on the fast variables.

From the viewpoints of methodology and applications it is of interest to develop an asymptotic procedure for the approximate solution of optimal control problems with a non-fixed instant of the process termination (time-optimal type problems). However, this problem requires a separate investigation due to the essential non-uniqueness of the solution of the boundary-value problem of the maximum principle: the number of roots of the transversality equation may be of an order of ε^{-1} [2, 3].

5. GENERALIZATION OF THE CONTROL PROBLEM

We will consider the following system of equations, which is more general compared with (1.1)

$$\begin{aligned} \ddot{q} &= Q(\theta, t, q, \dot{q}, z, u, \lambda), \quad q(t_0) = q^0, \quad \dot{q}(t_0) = \dot{q}^0 \\ \dot{z} &= Z_*(\theta, t, q, \dot{q}, z, u, \lambda^{-1}), \quad z(t_0) = z^0 \end{aligned} \tag{5.1}$$

The functions Q and Z_* are smooth with respect to the time t and the k-vector z, $z \in D_z$. The dependence on the other arguments is similar to that described above for the function Q (see Section 1). Boundary conditions of the type (1.2) and the functional (1.3) are taken in the form

$$\begin{aligned} &M(q, \dot{q}, z)|_T = 0, \quad M = (M_1, \dots, M_m), \quad 0 \le m \le 2n + k \\ &J[u] = g(q, \dot{q}, z)|_T + \int_{t_0}^{T} G(\theta, t, q, \dot{q}, z, u)dt \to \min_u, \quad u \in U \end{aligned} \tag{5.2}$$

The explicit introduction of t into the functions Q, Z and G is made for convenience; without loss of generality the argument t can be included in the vector z: $\dot{z}_{k+1} = 1$, $z_{k+1}(t_0) = t_0$.

By changing to the argument $\theta = \lambda t$ we obtain the following system of equations and initial data

$$q'' = \varepsilon^2 Q, \quad z' = \varepsilon Z_*; \quad q(\theta_0) = q^0, \quad q'(\theta_0) = \varepsilon \dot{q}^0, \quad z(\theta_0) = z^0 \tag{5.3}$$

The right-hand sides of Eqs (5.3) contain an irregular dependence on q' of the form $\varepsilon^{-1}q'$ in terms of the argument $\dot{q}$, see (2.1). With respect to the function Q it is required that the following condition, which generalizes (2.2), should be satisfied

$$\varepsilon^2 Q(\theta, t, q, \varepsilon^{-1}q', z, u, \lambda) \equiv \varepsilon R(\theta, t, q, q', z) + \varepsilon^2 S(\theta, t, q, \varepsilon^{-1}q', z, u, \varepsilon) \tag{5.4}$$

A possible regular dependence of the function R on the small parameter ε is not indicated; it may be related to the term $O(\varepsilon^2)$, i.e. to $\varepsilon^2 S$. It is assumed that the function R possesses zero mean with respect to θ over the period 2π when $q' = 0$. Using a change of variables [1] of the form $(q, q', z) \to (x, y, z)$ by differentiation of the explicit replacement formulae with respect to θ, by means of system (5.3), (5.4) we obtain the equations of the controlled motion in a standard form

$$\begin{aligned} x' &= \varepsilon X(\theta, t, x, y, z, u, \varepsilon), \quad x(\theta_0) = q^0, \quad t = \varepsilon\theta, \quad x \in D_q, \quad 0 < \varepsilon \ll 1 \\ y' &= \varepsilon Y(\theta, t, x, y, z, u, \varepsilon), \quad y(\theta_0) = \dot{q}^0, \quad (y + \Delta R^*) \in D_{\dot{q}} \\ z' &= \varepsilon Z(\theta, t, x, y, z, u, \varepsilon), \quad z(\theta_0) = z^0, \quad z \in D_z, \quad u \in U \end{aligned} \tag{5.5}$$

Unlike the expression for X (2.5) we also have a weak (of the order of ε) dependence on the control u and on the slow variables t and z. The relation between the variables and the right-hand sides of Eqs (5.5) are given by relations similar to (2.4),

$$\begin{aligned} &q = x + \varepsilon R^{**}(\theta, t, x, z), \quad q' = \varepsilon y + \varepsilon \Delta R^*(\theta, t, x, z), \quad z \equiv z \\ &R^{**} = \int_{\theta_0}^{\theta} \Delta R^* d\theta_1, \quad \Delta R^* = R^* - \langle R^* \rangle, \quad R^* = \int_{\theta_0}^{\theta} R_0 d\theta_1 \end{aligned} \tag{5.6}$$

in which the function R_0 is represented in terms of R (5.4) when $q' = 0$. The integration with respect to θ (5.6) is carried out for fixed t, x and z. Differentiation of the replacement formulae (5.6) leads to the required expressions for the functions X, Y and Z in (5.5)

$$\begin{aligned} X &= (I+\varepsilon R_x^{**\prime})^{-1}(y-\varepsilon R_t^{**\prime}-\varepsilon R_z^{**\prime}Z) = y+O(\varepsilon) \\ Y &= (R_x')R^{**}+(R_{q'}')(y+\Delta R^*)+\varepsilon(S)-\Delta R_x^{*\prime}y-\Delta R_z^{*\prime}(Z)+O(\varepsilon) \\ Z &= Z_*(\theta,t,x+\varepsilon R^{**},y+\Delta R^*,u,\varepsilon) = (Z)+O(\varepsilon) \\ S &= S(\theta,t,x+\varepsilon R^{**},y+\Delta R^*,u,\varepsilon) = (S)+O(\varepsilon) \end{aligned} \tag{5.7}$$

Expressions of the type (R_x'), (S), (Z) imply that the arguments of the functions are taken for $\varepsilon = 0$ and $q = x$. It follows from expressions (5.7) that $(X) = y$ while (Y) is the function written ignoring the term $O(\varepsilon)$.

The final conditions (5.2) and the functional (5.3) are transformed by means of the replacement (5.6). We obtain expressions similar to (2.7) and (2.8) respectively. As a result we obtain a problem of optimal terminal control over an asymptotically long interval of variation of the fast phase (argument) $t_0\varepsilon^{-1} = \theta_0 \le \theta \le \Theta = T\varepsilon^{-1}$ for the $(2n + k)$-dimensional vector of the osculating variables (x, y, z). The asymptotic procedure for the approximate solution [2, 3], described in Sections 3 and 4, can be applied to this problem. To estimate the accuracy, an assertion similar to the theorem stated above holds.

The approximate solution of control problems of time-optimal type requires a separate consideration.

6. EXAMPLES

We will consider one-dimensional rotatory-oscillatory systems, to which we will apply the above-mentioned asymptotic method. As a result of an asymptotic analysis we can construct much simpler models of controlled systems, not containing an explicit dependence on time and which enable us to use standard procedures (analytical and numerical).

1. *The control of plane oscillations and rotations of a rigid body about a rapidly vibrating axis.* The equation of motion of the body has the form [1–3]

$$A\ddot{\varphi}+\mu g l\sin\varphi = -\mu l(\ddot{\xi}_0(\nu t)\cos\varphi+\ddot{\eta}_0(\nu t)\sin\varphi)+V \tag{6.1}$$

with the corresponding initial data φ^0, $\dot{\varphi}^0$. Here φ is the angular coordinate, A is the moment of inertia about the axis, μ is the mass, l is the "arm" of the mass forces, the acceleration of which is equal to g, ξ_0, η_0 are the coordinates of the axis that performs plane-parallel motion, ν is the oscillation frequency and V is the moment of the external forces. By introducing the argument $\theta = \nu t$ we can represent Eq. (6.1) in the standard form (2.1), (2.2)

$$\begin{gathered} \varphi'' = -\varepsilon(\xi''(\theta)\cos\varphi+\eta''(\theta)\sin\varphi)-\varepsilon^2\kappa\sin\varphi+\varepsilon^2 u \\ \varepsilon = \frac{\mu l\rho}{A},\quad \varepsilon^2\kappa = \frac{\mu g l}{A\nu^2},\quad \xi = \frac{\xi_0}{\rho},\quad \eta = \frac{\eta_0}{\rho},\quad \varepsilon^2 u = \frac{V}{A\nu^2} \end{gathered} \tag{6.2}$$

where ρ is the amplitude of the oscillations of the axis. The initial data φ^0, $\dot{\varphi}^0$ for Eq. (6.1) and φ^0 and $\varepsilon\varphi'^0$ for Eq. (6.2) can correspond to an oscillatory or rotatory mode. The first term on the right-hand side has the meaning of the expression εR, independent of φ', while the second and third have the meaning of $\varepsilon^2 S$. We will assume that the functions ξ'', η'' have zero mean, i.e. condition (2.3) is satisfied. Then, by means of the change of variables (2.4) we obtain the following standard controlled system of the first approximation

$$\begin{aligned} x' &= \varepsilon y,\quad \varphi = x+\varepsilon R^{**},\quad x(0) = \varphi^0 \\ y' &= \varepsilon y(-\xi'\sin x+\eta'\cos x)+\varepsilon(\xi''\sin x-\eta''\cos x)R^{**}-\varepsilon\kappa\sin x+\varepsilon u \\ \varphi' &= \varepsilon y+\varepsilon\Delta R^*,\quad y(0) = \varphi'^0-\Delta R^*(0,x^0) \end{aligned} \tag{6.3}$$

The Hamilton function $H = \varepsilon H_0$ for controlled system (6.3) with a functional of the form (2.8) enables us, after maximization with respect to $u \in U$, to determine the optimal control u^* and the averaged Hamiltonian of the first approximation h in slow time $\tau = \varepsilon\theta$

$$\begin{aligned} u^* &= u(\theta, x, y, p_y), \quad \upsilon = \upsilon(x, y, p_y) = \langle u(\theta, x, y, p_y)\rangle \\ h(x, y, p_x, p_y) &= p_x y + p_y(-y\langle \Delta R_x^{*\prime}\rangle + \langle R_x' R^{**}\rangle - \kappa \sin x) + \upsilon \end{aligned} \tag{6.4}$$

Here $p_{x,y}$ are the momenta and υ is a new (averaged) control, which is used in the averaged problem. The mean values of the functions in relations (6.4) are calculated in explicit form

$$\langle \Delta R_x^{*\prime}\rangle = 0, \quad \langle R_x' R^{**}\rangle = \frac{1}{2}(\langle \xi'^2\rangle - \langle \eta'^2\rangle)\sin 2x - \langle \xi'\eta'\rangle \cos 2x \tag{6.5}$$

It is interesting to note that the first expression of (6.5) vanishes when $\langle \xi'\rangle$ or/and $\langle \eta'\rangle$ are non-zero, i.e. in the new variables there is no uniform on the average displacement of the axis. Analysis of the second expression is also of interest [1]. This term is related to the effect of the occurrence of other equilibrium positions (in addition to $x = 0, \pi$) and their vibrational stabilization in the uncontrolled system (when $u = \upsilon \equiv 0$).

The averaged control problem obtained is more sophisticated and difficult to investigate compared with the control problem for a physical pendulum with a fixed axis [1–3]. With certain simplifying assumptions it can be reduced to the problem indicated but with a different content. In particular, when $\kappa \sim \varepsilon$, for motion in an ellipse we have the averaged control problem

$$\begin{aligned} &\xi = \cos\gamma\cos\theta, \quad \eta = \sin\gamma\sin\theta, \quad 0 \le \gamma \le \pi \\ &x' = y, \quad y' = \frac{1}{4}\cos 2\gamma \sin 2x + \upsilon \end{aligned} \tag{6.6}$$

As a result we have obtained the system of equations (6.6) of the controlled motion for an equivalent pendulum with four positions of relative equilibrium $x = 0, \pi/2, \pi$ and $3\pi/2$ for $0 \le x < 2\pi$. The motion of the axis in a circle ($\gamma = \pi/4, 3\pi/4$) leads to a simple equation of the controlled motion $x'' = \upsilon$.

If the axis oscillates in a plane in a plane-parallel manner, then, by analogy with problem (6.6) we obtain

$$\begin{aligned} &\xi = \cos\gamma\cos\theta, \quad \eta = \sin\gamma\cos\theta, \quad \gamma = \text{const} \\ &x' = y, \quad y' = \frac{1}{4}\sin 2(x - \gamma) + \upsilon \end{aligned} \tag{6.7}$$

The control problems for systems (6.6) and (6.7) can be investigated by standard regular methods. In the general case, system (6.4), (6.5) can be reduced to the form

$$x' = y, \quad y' = ay\sin(x + \alpha) + b\sin(2x + \beta) - \kappa \sin x + \upsilon \tag{6.8}$$

where a, α, b and β are constants, determined by relations (6.5). To solve problems of controlling the motion of system (6.8) with various final conditions and functionals it is necessary to develop numerical methods, since further simplification is difficult.

2. *The controlled drift of a "microparticle" in the force field of a travelling wave.* In plasma theory it is of some interest to investigate the dynamics of "quasi-particles" in an alternating field, which is modelled by travelling or standing waves [8], or a wave packet [9].

With appropriate assumptions regarding the quasi-stationarity, the motion of the particle in a travelling-wave field is described by the equation [9] and initial conditions

$$m\ddot{s} = eE_0 \sin(ks - \omega t) + eE_1, \quad s(0) = s^0, \quad \dot{s}(0) = \dot{s}^0 \tag{6.9}$$

Here m is the mass and e is the charge of the particle, E_0 is the amplitude of the intensity of the travelling wave, k is the wave number and ω is the frequency of the oscillations. Equation (6.9) contains the additional term eE_1, which has the meaning of a small control function.

We will introduce dimensionless variables and parameters in the required way; we obtain a controlled system of the form (2.1), (2.2)

$$
\begin{aligned}
& q'' = \varepsilon\sin(q-\theta)+\varepsilon^2 u, \quad q(0) = q^0, \quad q'(0) = \varepsilon q'^0 \\
& q = ks, \quad q' = k\dot{s}/\omega, \quad \theta = \omega t, \quad q^0, q'^0 \sim 1 \\
& \varepsilon = eE_0k/(m\omega^2) \ll 1, \quad \varepsilon^2 u = eE_1k/(m\omega^2), \quad u \in U
\end{aligned} \tag{6.10}
$$

According to system (6.10) the intensity E_1 of the control field is an order of magnitude less with respect to the small parameter ε than the amplitude E_0 of the travelling wave. After transformation (2.4) we obtain, in the first approximation in ε, the following controlled system of standard form (2.5)

$$
\begin{aligned}
& x' = \varepsilon y, \quad x(0) = q^0 \equiv x^0 \\
& y' = \varepsilon y\sin(x-\theta)+\varepsilon\cos(x-\theta)[\sin x-\sin(x-\theta)]+\varepsilon u, \quad y(0) = q'^0-\cos q^0 \equiv y^0 \\
& q = x+\varepsilon[\sin x-\sin(x-\theta)], \quad q' = \varepsilon y+\varepsilon\cos(x-\theta)
\end{aligned} \tag{6.11}
$$

From the applied point of view, it is of interest to bring the phase point $(s, \dot{s})$ of system (6.9) at a certain instant of time T to the origin of coordinates (0, 0), i.e. $q(\Theta) = q'(\Theta) = 0$. Using the asymptotic procedure described in Sections 3 and 4, the variable q is brought to the ε-neighbourhood while q' is brought to the ε^2-neighbourhood of the required value. We recall that the variable $q' - O(\varepsilon)$.

The averaged equations and the final conditions (when $\theta = \Theta = \omega T$) of the first approximation, after introducing the slow argument $\tau = \varepsilon\theta$, according to system (6.11) take the form

$$
\begin{aligned}
& \dot{x} = y, \quad \dot{y} = v, \quad 0 \le \tau \le \tau_f, \quad v = \langle u^* \rangle \\
& x^f = x(\tau_f) = 0, \quad y^f = y(\tau_f) = -\cos\Theta
\end{aligned} \tag{6.12}
$$

Suppose, in the functional $J[u]$ (2.8), the functions $g \equiv 0$, $G = u^2/2$; then $u^* = p_y = -p_x^0\tau + p_y^0$ is a linear function of τ. The momentum $p_x = p_x^0$ is constant in the first approximation; for constant $p_{x,y}^0$, after integration of the equations, from the final conditions (6.12) we obtain the expressions

$$
\begin{aligned}
& p_x^0 = 6\tau_f^{-3}(2\Delta x^f - \Delta y^f\tau_f), \quad \Delta x^f = x^f-(x^0+y^0\tau_f) \\
& p_y^0 = 2\tau_f^{-2}(3\Delta x^f - 2\Delta y^f\tau_f), \quad \Delta y^f = y^f - y^0
\end{aligned} \tag{6.13}
$$

Note that in the averaged Hamiltonian the periodic terms of Eq. (6.11) make no contribution, i.e. the travelling wave does not lead to drift of the "particles" in the first approximation. The control u^* is a linear function of the slow argument $\tau = \varepsilon\theta$, see (6.10), with coefficients $p_{x,y}^0$ (6.13). The change of the slow variables $\tau \to 0$, $\tau_f \to \tau_f - \tau$, $x^0 \to x$, $y^0 \to y$ reduces this control to a feedback form (the fast variable θ and the quantity Θ are not subject to transformations), the feedback being negative.

Note that when constructing the approximate asymptotic solution the contradictory requirement that the wave should be quasi-stationary and high-frequency ($\omega \to \infty$) is unimportant. What is important is that the normalization condition (6.10) should be satisfied, and this is achieved by appropriate assumptions.

A controlled system of the form (6.9) can be obtained using model (6.1) assuming $g = 0$ and oscillations of the axis

$$
\xi = -\rho\sin\nu t, \quad \eta = \rho\cos\nu t, \quad \rho = \text{const}
$$

which corresponds to its motion in a circle of radius ρ (see (6.6)).

3. *Control of the evolution of "microparticles" in a standing-wave field.* We will consider the modified problem [8] for the case of similar oppositely travelling waves. Unlike problem (6.9), the first term on the right-hand side of the equation of the controlled motion has the form

$$
eE_0[\sin(ks-\omega t)+\sin(ks+\omega t)]
$$

The dimensionless variables θ, q, q' and the control $\varepsilon^2 u$ are introduced according to relations (6.10), while the small parameter ε is more conveniently introduced as follows: $\varepsilon = 2eE_0k/(m\omega^2)$. The equation for q takes the form

$$q'' = \varepsilon \sin q \cos\theta + \varepsilon^2 u, \quad q(0) = q^0, \quad q'(0) = \varepsilon q'^0 \tag{6.14}$$

We apply the asymptotic procedure described in Sections 2–4 to system (6.14). By means of transformation (2.4) we obtain a standard controlled system of the first approximation of type (2.5)

$$\begin{aligned} &x' = \varepsilon y, \quad x(0) = q^0 \\ &y' = -\varepsilon y \sin\theta \cos x + \frac{1}{2}\varepsilon \sin 2x \cos\theta(1-\cos\theta) + \varepsilon u, \quad y(0) = q^0 \\ &q = x + \varepsilon \sin x(1-\cos\theta), \quad q' = \varepsilon y + \varepsilon \sin x \sin\theta \end{aligned} \tag{6.15}$$

The use of the averaging procedure leads to a controlled system (6.15) of the pendulum type, similar to (6.8),

$$x^{\cdot} = y, \quad y^{\cdot} = -\frac{1}{4}\sin 2x + \upsilon, \quad \upsilon = \langle u^* \rangle \tag{6.16}$$

As a result we have the problem of constructing a "control" $\upsilon(x, y, p_y)$ (see (6.4)), on the basis of the solution of the boundary-value problem for the averaged variables x and y (6.16) and the momenta p_x and p_y

$$p_x^{\cdot} = \frac{1}{2}p_y \cos 2x, \quad p_y^{\cdot} = -p_x$$

with corresponding final conditions and transversality conditions, in particular $\upsilon = p_y$ when $G = u^2/2$, $|u| < \infty$.

Hence, the problem of controlling the evolution of the modified system differs considerably from problem (6.9), considered above, since in the first approximation it contains a "restoring force" – $\sin(2x)/4$. This fact seems paradoxical, since the forward and backward waves occur additively and independently have no effect on the particle drift in the first approximation. However, their combined action leads to this effect, due to the fact that the first approximation contains quadratic terms (see (2.5)). In the case considered, non-zero averaged expressions on the right-hand side of the equation for y appear due to the mutual effect of the forward and backward waves.

An investigation of the dynamics and controlled drift of "microparticles" in a alternating field may be of interest from the viewpoints of methodology and applications. It can be modelled by a wave packet [9] and taking the perturbing factors (the non-stationarity of the parameters, the resistance of the medium, etc.) into account. With standard assumptions the procedure described in Sections 2–5 can be applied to these problems. However, their investigation requires a separate discussion. Note that force actions of the travelling-wave, oppositely travelling wave or wave-packet types can be achieved by means of a model of an oscillating and rotating body with periodically or quasi-periodically moving axis.

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